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Solutions of a two-dimensional Skyrme model on Aff($\mathbb{R}$)

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Abstract

No explicit solutions of the Skyrme field equation have been found since its formulation, more than 35 years ago. Here we reduce this equation to a two-dimensional model, for which we obtain a whole class of solutions. Therefore, we change the three-dimensional SU(2)-valued field to a two-dimensional Aff($\mathbb{R}$)-valued field, where Aff($\mathbb{R}$) is the only two-dimensional, connected, non-commutative Lie group. We will give examples for this reduced field equation.

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1. Introduction

The Skyrme model is a non-linear field theory of low energy hadron physics, which identifies baryons as topological solitons of a field theory for pions. It was introduced by the British physicist T.H.R. Skyrme (1922–87) in early 1960s and found its revival in its context to Quantum Chromodynamics (QCD).

Using Goldstone boson fields $(\pi^0, \pi^1, \pi^2) : M^4 \to \mathbb{R}$ on space–time M^4 we may write a Skyrme field

$$U : M^4 \to \mathrm{SU}(2) \quad \text{as}\, U = \exp(\mathrm{i}C),$$

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where

$$C := \begin{pmatrix} \pi^0 & \pi^1 + \mathrm{i}\pi^2 \\ \pi^1 - \mathrm{i}\pi^2 & -\pi^0 \end{pmatrix} = \begin{pmatrix} \pi^0 & \pi^+ \\ \pi^- & -\pi^0 \end{pmatrix} \tag{1}$$

with $\pi^\pm := \pi^1 \pm \pi^2 : M^4 \to \mathbb{C}$. To ensure finite energy, the Goldstone boson fields $\pi^a(\boldsymbol{x})$ ($a = 0, 1, 2$) are required to tend to zero and thus $U(\boldsymbol{x}) \to 1$ for $r(\boldsymbol{x}) \to \infty$, where $r = (x^2 + y^2 + z^2)^{-1/2}$ denotes the spatial radius of $\boldsymbol{x} \in M^4$ (cf. [1]).

The Lagrangian density of the Skyrme model is given by (note that we use Einstein summation convention)

$$\mathcal{L} = -\frac{F_\pi^2}{16}\mathrm{Tr}(L_\mu L^\mu) + \frac{1}{32g^2}\mathrm{Tr}([L_\mu, L_\nu][L^\mu, L^\nu]),$$

where $L_\mu := L(e_\mu)$ are the components of the Maurer–Cartan form $L = U^{-1}\,\mathrm{d}U$ ($\mu = 0, 1, 2, 3$), $\{e_\mu\}$ local coordinates on M^4 and indices are raised using the Minkowski metric $(+, -, -, -)$ of the space–time M^4. The constants are the pion decay constant F_π and a unit-free parameter g introduced by Skyrme to stabilize the solitons [2,3]. The values for F_π and g can be found in [3]. Without loss of generality, we will fix $F_\pi \cdot g = 1$. This leads to the Skyrme field equation [2]

$$0 = \partial_\mu \left(L^\mu + \tfrac{1}{4}\Big[L_\nu, [L^\nu, L^\mu]\Big]\right) = -\delta\left(L + \tfrac{1}{4}\Big[L(e_\nu), [L(\mathrm{e}^\nu), L]\Big]\right) \tag{2}$$

with co-derivative $\delta\omega_k := (-1)^k * \mathrm{d}{*}\omega_k$ for k-forms ω_k, where $*$ denotes the Hodge star operator on M^4.

In this paper we will reduce the field equation to a two-dimensional model, i.e., we are looking at maps

$$U: M \longrightarrow \mathrm{Aff}(\mathbb{R}),$$

which satisfy the Skyrme field equation (2), where M is an arbitrary connected, n-dimensional, pseudo-Riemannian manifold and $\mathrm{Aff}(\mathbb{R})$ is a two-dimensional Lie group. The goal is to find examples for these maps U.

To accomplish this, we will introduce the Lie group $\mathrm{Aff}(\mathbb{R})$ and its basic properties in Section 2. In Section 3 we will obtain two conditions for functions on M, which are equivalent to solving the field equation (2). Finally, in Section 4 we will give a class of examples.

2. The Lie group Aff ($\mathbb{R}$)

The Lie group $\mathrm{Aff}(\mathbb{R})$ can be defined as

$$\mathrm{Aff}(\mathbb{R}) = \left\{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a, b \in \mathbb{R}, a > 0\right\},$$

and its Lie algebra has the form

$$\mathrm{aff}(\mathbb{R}) = \left\{ \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} : x, y \in \mathbb{R} \right\} = \mathrm{Lin}_{\mathbb{R}}(\tau_1, \tau_2)$$

with a basis

$$\tau_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \tau_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The commutator relation of the Lie algebra $\mathrm{aff}(\mathbb{R})$ is given by $[\tau_1, \tau_2] = \tau_2$. Thus it is obvious that this Lie group is the only two-dimensional, non-commutative one. It is also a non-compact Lie group, which allows no Ad-invariant scalar product like the Killing form.

From the physicist's point of view, this model corresponds to neglecting the π^-- (resp., π^+-) excitations. In fact, we may write C in (1) as

$$C = 2\pi^0\tilde{\tau}_1 + \pi^+\tilde{\tau}_2 + \pi^-\tilde{\tau}_3 := 2\pi^0 \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} + \pi^+ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \pi^- \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then we obtain the relation $[\tilde{\tau}_1, \tilde{\tau}_2] = \tilde{\tau}_2$. Thus the Lie algebra generated by $\tilde{\tau}_1$ and $\tilde{\tau}_2$ is isomorphic to $\mathrm{aff}(\mathbb{R})$ – the case we describe here.

Since $\mathrm{Aff}(\mathbb{R})$ looks topologically like $\mathbb{R}^+ \times \mathbb{R}$, the kth homotopy group $\pi_k(\mathrm{Aff}(\mathbb{R}))$ is trivial for any k. Therefore, we have no chance to find a topological invariant for the field $U: M \to \mathrm{Aff}(\mathbb{R})$, i.e., this two-dimensional model has no topological quantum number like the classical SU(2)-Skyrme model, where this topological quantum number is interpreted as the number of baryons described by a given field U. Nevertheless, recall that these Skyrme fields have non-trivial winding numbers only if all boson fields are excited, and so the baryon number for the fields we are dealing with would be zero anyhow.

With regard to the Skyrme field equation (2), we are interested in the Maurer–Cartan form of $\mathrm{Aff}(\mathbb{R})$. So with

$$A = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in \mathrm{Aff}(\mathbb{R}),$$

$$A^{-1} = \begin{pmatrix} 1/a & -b/a \\ 0 & 1 \end{pmatrix}, \qquad \mathrm{d}A = \begin{pmatrix} \mathrm{d}a & \mathrm{d}b \\ 0 & 0 \end{pmatrix},$$

we obtain the Maurer–Cartan form

$$L := A^{-1}\,\mathrm{d}A = \begin{pmatrix} (1/a)\,\mathrm{d}a & (1/a)\,\mathrm{d}b \\ 0 & 0 \end{pmatrix} = \frac{1}{a}\,\mathrm{d}a \cdot \tau_1 + \frac{1}{a}\,\mathrm{d}b \cdot \tau_2 \in \mathrm{aff}(\mathbb{R}). \tag{3}$$

These are the basic facts we have to know for the following calculations.

3. Solutions of the field equation

In this section, we will transform the field equation into two conditions for functions on the manifold M.

Let us write the map $U: M \longrightarrow \mathrm{Aff}(\mathbb{R})$ in the general form

$$U(\boldsymbol{x}) = \begin{pmatrix} \mathrm{e}^{f(\boldsymbol{x})} & g(\boldsymbol{x}) \\ 0 & 1 \end{pmatrix} \tag{4}$$

with functions f, g on the manifold M, such that $f(\boldsymbol{x}), g(\boldsymbol{x}) \to 0$ for $r(\boldsymbol{x}) \to \infty$. Thus, we obtain from (3) for the Maurer–Cartan form

$$L = \mathrm{d}f \cdot \tau_1 + \mathrm{e}^{-f}\mathrm{d}g \cdot \tau_2.$$

Use of this information for the field equation (2) results in:

$$\begin{aligned}0 &= \delta\Big(L + \tfrac{1}{4}\Big[L(e_\nu), [L(\mathrm{e}^\nu), L]\Big]\Big)\\ &= \delta\Big(\mathrm{d}f \cdot \tau_1 + \mathrm{e}^{-f}\mathrm{d}g \cdot \tau_2\\ &\quad + \tfrac{1}{4}(\mathrm{d}f(e_\nu) \cdot \mathrm{d}f(\mathrm{e}^\nu)\mathrm{e}^{-f}\mathrm{d}g - \mathrm{d}f(e_\nu) \cdot \mathrm{d}g(\mathrm{e}^\nu)\mathrm{e}^{-f}\mathrm{d}f) \cdot \tau_2\Big).\end{aligned}$$

In order to avoid local coordinates $\{e_\nu\}$ we may use the Laplace operator $\Delta = \delta\,\mathrm{d} + \mathrm{d}\delta$ and the scalar product $\langle\ ,\ \rangle$ of differential forms, where $\langle \mathrm{d}h, \mathrm{d}k\rangle = \mathrm{d}h(e_\nu) \cdot \mathrm{d}k(\mathrm{e}^\nu)$ for functions h and k on M. Thus the field equation is equivalent to the following two conditions on f and g:

(i) $\Delta f = 0$ and

(ii) $\delta\left(\mathrm{e}^{-f}\,\mathrm{d}g + \frac{1}{4}\langle \mathrm{d}f, \mathrm{d}f\rangle \mathrm{e}^{-f}\,\mathrm{d}g - \frac{1}{4}\langle \mathrm{d}f, \mathrm{d}g\rangle \mathrm{e}^{-f}\,\mathrm{d}f\right) = 0.$

Using now the relation $\delta h\,\mathrm{d}k = -\langle \mathrm{d}h, \mathrm{d}k\rangle + h\Delta k$, we may rewrite condition (ii) as

$$\begin{aligned}&\langle \mathrm{d}f, \mathrm{d}g\rangle\left(1 + \tfrac{1}{4}\langle \mathrm{d}f, \mathrm{d}f\rangle\right) - \tfrac{1}{4}\langle \mathrm{d}f, \mathrm{d}f\rangle\langle \mathrm{d}f, \mathrm{d}g\rangle + (\Delta g)\left(1 + \tfrac{1}{4}\langle \mathrm{d}f, \mathrm{d}f\rangle\right)\\ &\quad - \tfrac{1}{4}\langle \mathrm{d}g, \mathrm{d}\langle \mathrm{d}f, \mathrm{d}f\rangle\rangle - \tfrac{1}{4}(\Delta f)\langle \mathrm{d}f, \mathrm{d}g\rangle + \tfrac{1}{4}\langle \mathrm{d}f, \mathrm{d}\langle \mathrm{d}f, \mathrm{d}g\rangle\rangle = 0,\end{aligned}$$

and with condition (i), $\Delta f = 0$, it reduces to

$$\begin{aligned}0 &= \langle \mathrm{d}f, \mathrm{d}g\rangle + (\Delta g)\left(1 + \tfrac{1}{4}\langle \mathrm{d}f, \mathrm{d}f\rangle\right) - \tfrac{1}{4}\langle \mathrm{d}g, \mathrm{d}\langle \mathrm{d}f, \mathrm{d}f\rangle\rangle\\ &\quad + \tfrac{1}{4}\langle \mathrm{d}f, \mathrm{d}\langle \mathrm{d}f, \mathrm{d}g\rangle\rangle. \qquad (5)\end{aligned}$$

Note that for a compact Riemannian manifold M, every harmonic function is constant, and with our boundary condition $f \to 0$ for $r \to \infty$ we obtain from $\Delta f = 0$ that $f \equiv 0$. This, in turn, results in $\Delta g = 0$ and thus we only get the trivial solution in this case.

So, we are looking for solutions on non-compact Riemannian manifolds and manifolds with a metric of index different from zero.

4. Examples for Skyrme fields

Example 1. Let us first look at the trivial case for f, namely $\mathrm{d}f \equiv 0$. As was said before, if f is constant, then our boundary condition on f requires $f \equiv 0$. In this case, condition (ii) is equivalent to $\Delta g = 0$. So every

$$U(x) = \begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix}$$

with harmonic g and $g \to 0$ for $r \to \infty$ solves the field equation. Of course, solutions of $\Delta g = 0$ are well-known, let us just mention some of them. We use spherical coordinates (r, θ, ϕ) on $\mathbb{R}^3$ and $\rho = (x^2 + y^2)^{-1/2}$. As static solutions we obtain

$$g(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} \frac{C_{lm}}{r^{l+1}} P_l^m(\cos\theta) \sin(m\phi + \phi_m) \quad \text{with } C_{lm}, \phi_m \in \mathbb{R}, \tag{6}$$

where P_l^m denote the associated Legendre polynomials in $\cos\theta$ defined by

$$P_l^m(\cos\theta) = \frac{(-1)^m}{2^l l!}(1 - \cos^2\theta)^{m/2} \frac{\mathrm{d}^{l+m}}{\mathrm{d}(\cos\theta)^{l+m}}\{(\cos^2\theta - 1)^l\}.$$

Recall that for $m = 0$ the first Legendre polynomials are

$$P_0(\cos\theta) = 1, \qquad P_1(\cos\theta) = \cos\theta, \qquad P_2(\cos\theta) = \tfrac{1}{2}(3\cos^2\theta - 1),$$

and we thus have the following linear independent solutions:

$$g_0 = k_0 \frac{1}{r}, \quad g_1 = k_1 \frac{z}{r^3}, \quad g_2 = k_2 \frac{3z^2 - r^2}{2r^5} \qquad \text{with } k_i \in \mathbb{R}.$$

In two dimensions, i.e., independent of time t and space z, the solution is

$$g(\rho, \phi) = \sum_{m=1}^{\infty} \frac{A_m}{\rho^m} \sin(m\phi + \phi_m) \quad \text{with } A_m, \phi_m \in \mathbb{R}. \tag{7}$$

Note that all these solutions have a singularity at $r = 0$, resp., $\rho = 0$. In fact, we will not find a non-trivial smooth solution without any singularity, because in that case, this would also be a harmonic function on the one-point compactification of the space manifold, and hence would be constant, i.e., zero.

As non-static solutions on Minkowski space we find

$$g(t, \rho, \phi, z) = G(t \pm z) \cdot g(\rho, \phi)$$

with $g(\rho, \phi)$ from (7) and any $G \in C^\infty(\mathbb{R})$, or more generally, with $G_m \in C^\infty(\mathbb{R})$,

$$g(t, \rho, \phi, z) = \sum_{m=1}^{\infty} G_m(t \pm z) \frac{A_m}{\rho^m} \sin(m\phi + \phi_m).$$

These solutions describe wave functions in the $\pm z$-direction, e.g., in order to ensure that for every time t, our boundary conditions are satisfied, G and the G_m may be taken to be gaussian. Generalizations to wave functions in any direction are immediate, e.g., any superposition of plane waves

$$g(t, \boldsymbol{r}) = \int_{\boldsymbol{k}} A_{\boldsymbol{k}} \sin(\boldsymbol{k}\boldsymbol{r} - |\boldsymbol{k}|t - \phi_{\boldsymbol{k}}) \,\mathrm{d}\boldsymbol{k}^3$$

is harmonic. Nevertheless, note that plane waves do not obey our boundary condition.

Example 2. We define

$$\omega := \left(\mathrm{e}^{-f}\,\mathrm{d}g + \tfrac{1}{4}\langle \mathrm{d}f, \mathrm{d}f\rangle \mathrm{e}^{-f}\,\mathrm{d}g - \tfrac{1}{4}\langle \mathrm{d}f, \mathrm{d}g\rangle \mathrm{e}^{-f}\,\mathrm{d}f\right),$$

i.e., condition (ii) is given by $\delta\omega = 0$. Hence

$$\langle \mathrm{d}f, \omega\rangle = \mathrm{e}^{-f}\langle \mathrm{d}f, \mathrm{d}g\rangle.$$

Now suppose we have $0 \le \langle \mathrm{d}f, \mathrm{d}f\rangle$, which, e.g., is always the case on a Riemannian manifold. Then $\left(1 + \frac{1}{4}\langle \mathrm{d}f, \mathrm{d}f\rangle\right)^{-1} > 0$ and, together with the definition of ω, this yields

$$\mathrm{d}g = \mathrm{e}^{f} \cdot \frac{\omega + \frac{1}{4}\langle \mathrm{d}f, \omega\rangle\,\mathrm{d}f}{1 + \frac{1}{4}\langle \mathrm{d}f, \mathrm{d}f\rangle}. \tag{8}$$

If the right-hand side was an exact 1-form, or at least closed (and M simply connected), then this would be an equivalent condition to the second. Unfortunately, this is not the case in general, but leads to a condition similar to (5). Nevertheless there is a simple example, where exactness can be checked immediately: Let $\omega = c \cdot \mathrm{d}f$ for any constant $c \in \mathbb{R}$. Then (8) yields $\mathrm{d}g = c \cdot \mathrm{e}^{f}\,\mathrm{d}f$, i.e., $g = c \cdot \mathrm{e}^{f} + \tilde{c}$, with constant $\tilde{c} \in \mathbb{R}$. Our boundary conditions require $\tilde{c} = -c$. Thus a second example for a class of fields which solve the field equation (2) is

$$U(x) = \begin{pmatrix} \mathrm{e}^{f} & c \cdot (\mathrm{e}^{f} - 1) \\ 0 & 1 \end{pmatrix}$$

for any harmonic function f on M and constant $c \in \mathbb{R}$. Examples for f can easily be found analogously to Example 1.

Example 3. Finally let us discuss the case, where $\mathrm{d}f \neq 0$, but $\langle \mathrm{d}f, \mathrm{d}f\rangle = 0$ in a pseudo-Riemannian manifold, e.g., the Minkowski space. An obvious solution of (5) is $\langle \mathrm{d}f, \mathrm{d}g\rangle = 0$ and $\Delta g = 0$. Note that if g is such a solution, then also $g \cdot \mathrm{e}^{f}$ is a solution of (5). More generally, if g is harmonic and obeys $\langle \mathrm{d}f, \mathrm{d}g\rangle = 0$, then the same holds for

$$g' := g \cdot F_1(f) + F_0(f) \quad \text{with } F_0, F_1 \in C^{\infty}(\mathbb{R}),$$

e.g., if $\mathrm{d}g = h\,\mathrm{d}f$ with any smooth map h, then $\langle \mathrm{d}f, \mathrm{d}g\rangle = h \cdot \langle \mathrm{d}f, \mathrm{d}f\rangle = 0$ and $\Delta g = -\langle \mathrm{d}h, \mathrm{d}f\rangle = 0$, because $0 = \mathrm{d}^2 g = \mathrm{d}h \wedge \mathrm{d}f$, i.e., $\mathrm{d}h$ and $\mathrm{d}f$ are co-linear. Note that this also yields that h is harmonic. Examples for this situation are numerous, let us only state in $\mathbb{R} \times \mathbb{R}$:

$$f(t, z) = F(t \pm z), \quad g(t, z) = G(t \pm z) \quad \text{with } F, G \in C^{\infty}(\mathbb{R}).$$

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